

Relative symplectic Steinberg group

Andrei Lavrenov*

Abstract

We give two definitions of relative symplectic Steinberg group and show that they coincide.

Introduction

The main result of the present paper is a “relative version” of the *another presentation* for symplectic Steinberg groups obtained in [2].

First, we give a definition of a relative symplectic group $\mathrm{StSp}_{2l}(R, I, \Gamma)$ for any form ideal (I, Γ) in R . For maximal form parameter $\Gamma = I$ it has already appeared in [3] and is shown to have good homological properties. We also consider case of a maximal form parameter and establish another “coordinate-free” presentation for $\mathrm{StSp}_{2l}(R, I, \Gamma_{\max}) = \mathrm{StSp}_{2l}(R, I)$. The last result is a relativisation of the [2], but it is not a generalisation, since we need more relations for the presentation. Moreover, in our proofs we use the results of [2].

Namely, the main theorem of [2] states the following.

Theorem. *Let $l \geq 3$, then the symplectic Steinberg group $\mathrm{StSp}(2l, R)$ can be defined by the set of generators*

$$\left\{ X(u, v, a) \mid u, v \in V, \text{ } u \text{ is a column of } \right. \\ \left. \text{a symplectic elementary matrix, } \langle u, v \rangle = 0, a \in R \right\}$$

*The author acknowledges support of the RSCF project 14-11-00297 “Decomposition of unipotents in reductive groups”.

and relations

$$X(u, v, a)X(u, w, b) = X(u, v + w, a + b + \langle v, w \rangle), \quad (\text{P1})$$

$$X(u, va, 0) = X(v, ua, 0) \text{ where } v \text{ is also a column} \\ \text{of a symplectic elementary matrix,} \quad (\text{P2})$$

$$X(u', v', b)X(u, v, a)X(u', v', b)^{-1} = \\ = X(T(u', v', b)u, T(u', v', b)v, a), \quad (\text{P3})$$

where $T(u, v, a)$ is an ESD-transformation

$$w \mapsto w + u(\langle v, w \rangle + a\langle u, w \rangle) + v\langle u, w \rangle.$$

For usual generators of the symplectic Steinberg group the following identities hold

$$X_{ij}(a) = X(e_i, e_{-j}a\varepsilon_{-j}, 0) \text{ for } j \neq -i, \quad X_{i,-i}(a) = X(e_i, 0, a).$$

We prove the following theorem in the case of maximal form parameter.

Theorem 1. *Let $l \geq 3$, then the relative symplectic Steinberg group $\text{StSp}_{2l}(R, I)$ can be defined by the set of generators*

$$\{(u, v, a, b) \mid u, v \in V, u \text{ is a column of} \\ \text{a symplectic elementary matrix, } \langle u, v \rangle = 0, a, b \in I\}$$

and relations

$$(u, vr, a, b) = (u, v, ar, b) \text{ for any } r \in R, \quad (\text{T1})$$

$$(u, v, a, b)(u, w, a, c) = (u, v + w, a, b + c + a^2\langle v, w \rangle), \quad (\text{T2})$$

$$(u, v, a, 0)(u, v, b, 0) = (u, v, a + b, 0), \quad (\text{T3})$$

$$(u, v, a, 0) = (v, u, a, 0) \text{ for } v \text{ a column of} \\ \text{a symplectic elementary matrix,} \quad (\text{T4})$$

$$(u', v', a', b')(u, v, a, b)(u', v', a', b')^{-1} = \\ = (T(u', v'a', b')u, T(u', v'a', b')v, a, b), \quad (\text{T5})$$

$$(u, u, a, 0) = (u, 0, 0, 2a), \quad (\text{T6})$$

$$(u + vr, 0, 0, a) = (u, 0, 0, a)(v, 0, 0, ar^2)(v, u, ar, 0) \text{ for } v, u + vr \\ \text{also columns of symplectic elementary matrices.} \quad (\text{T7})$$

and for usual relative generators the following identities hold

$$Y_{ij}(a) = (e_i, e_{-j}, a\varepsilon_{-j}, 0) \text{ for } j \neq -i, \quad Y_{i,-i}(a) = (e_i, 0, 0, a).$$

The author plans to generalise the above result for the case of arbitrary form parameter.

1 Relative symplectic Steinberg group

In the sequel R denotes an arbitrary associative commutative unital ring, $V = R^{2l}$ denotes a free right R -module with basis numbered $e_{-l}, \dots, e_{-1}, e_1, \dots, e_l$, $l \geq 3$. For the vector $v \in V$ its i -th coordinate will be denoted by v_i , i.e. $v = \sum_{i=-l}^l e_i v_i$. By $\langle \cdot, \cdot \rangle$ we denote the standard symplectic form on V , i.e. $\langle e_i, e_j \rangle = \text{sgn}(i)\delta_{i,-j}$. We will usually write ε_i instead of $\text{sgn}(i)$. Observe that $\langle u, u \rangle = 0$ for any $u \in V$.

Definition. Define the *symplectic group* $\text{Sp}(V) = \text{Sp}_{2l}(R)$ as the group of automorphisms of V preserving the symplectic form $\langle \cdot, \cdot \rangle$,

$$\text{Sp}(V) = \{f \in \text{GL}(V) \mid \langle f(u), f(v) \rangle = \langle u, v \rangle \quad \forall u, v \in V\}.$$

Definition (Eichler–Siegel–Dickson transformations). For $a \in R$ and $u, v \in V$, $\langle u, v \rangle = 0$, denote by $T(u, v, a)$ the automorphism of V s.t. for $w \in V$ one has

$$T(u, v, a): w \mapsto w + u(\langle v, w \rangle + a\langle u, w \rangle) + v\langle u, w \rangle.$$

We refer to the elements $T(u, v, a)$ as the (*symplectic*) *ESD-transformations*.

Lemma 1. Let $u, v, w \in V$ be three vectors such that $\langle u, v \rangle = 0$, $\langle u, w \rangle = 0$, and let $a, b \in R$. Then

- a) $T(u, v, a) \in \text{Sp}(V)$,
- b) $T(u, v, a)T(u, w, b) = T(u, v + w, a + b + \langle v, w \rangle)$,
- c) $T(u, va, 0) = T(v, ua, 0)$,
- d) $gT(u, v, a)g^{-1} = T(gu, gv, a) \quad \forall g \in \text{Sp}(V)$.

Remark. Observe that $T(u, 0, 0) = 1$ and $T(u, v, a)^{-1} = T(u, -v, -a)$.

In the present paper all commutators are left-normed, $[x, y] = xyx^{-1}y^{-1}$, we denote xyx^{-1} by ${}^x y$.

Lemma 2. For $u, v \in V$ such that $u_i = u_{-i} = v_i = v_{-i} = 0$, $\langle u, v \rangle = 0$, and $a \in R$ one has

$$[T(e_i, u, 0), T(e_{-i}, v, a)] = T(u, v\varepsilon_i, a)T(e_{-i}, -ua\varepsilon_{-i}, 0).$$

Definition. A *form ideal* (I, Γ) is a pair of an ideal $I \leq R$ and a *relative form parameter* Γ of level I , i.e., an additive subgroup of I such that

- a) $\forall a \in I$ holds $2a \in \Gamma$,
- b) $\forall r \in R, \forall a \in I$ holds $ra^2 \in \Gamma$,
- c) $\forall \alpha \in \Gamma, \forall r \in R$ holds $\alpha r^2 \in \Gamma$.

Remark. If $2 \in R^\times$ the only possible choice for a relative form parameter is $\Gamma = I$.

Definition. Define $T_{ij}(a) = T(e_i, e_{-j}a\varepsilon_{-j}, 0)$ and $T_{i,-i}(a) = T(e_i, 0, a)$, where $a \in R, i, j \in \{-l, \dots, -1, 1, \dots, l\}, i \notin \{\pm j\}$. We refer to these elements as the *elementary symplectic transvections*. A normal subgroup of $\text{Sp}(V)$

$$\text{Ep}_{2l}(R, I, \Gamma) = \text{Sp}(V) \langle T_{ij}(a), T_{i,-i}(\alpha) \mid i \notin \{\pm j\}, a \in I, \alpha \in \Gamma \rangle$$

is called the *relative elementary symplectic group* corresponding to the form ideal (I, Γ) .

Definition. The *symplectic Steinberg group* $\text{StSp}_{2l}(R)$ is the group generated by the formal symbols $X_{ij}(r), i \neq j, r \in R$ subject to the Steinberg relations

$$X_{ij}(r) = X_{-j,-i}(-r\varepsilon_i\varepsilon_j), \quad (\text{S0})$$

$$X_{ij}(r)X_{ij}(s) = X_{ij}(r+s), \quad (\text{S1})$$

$$[X_{ij}(r), X_{hk}(s)] = 1, \text{ for } h \notin \{j, -i\}, k \notin \{i, -j\}, \quad (\text{S2})$$

$$[X_{ij}(r), X_{jk}(s)] = X_{ik}(rs), \text{ for } i \notin \{-j, -k\}, j \neq -k, \quad (\text{S3})$$

$$[X_{i,-i}(r), X_{-i,j}(s)] = X_{ij}(rs\varepsilon_i)X_{-j,j}(-rs^2), \quad (\text{S4})$$

$$[X_{ij}(r), X_{j,-i}(s)] = X_{i,-i}(2rs\varepsilon_i). \quad (\text{S5})$$

The next lemma is a straightforward consequence of Lemmas 1 and 2.

Lemma 3. *There is a natural epimorphism*

$$\phi: \text{StSp}_{2l}(R) \twoheadrightarrow \text{Ep}_{2l}(R) = \text{Ep}_{2l}(R, R, R)$$

sending the generators $X_{ij}(a)$ to the corresponding elementary transvections $T_{ij}(a)$. In other words, the Steinberg relations hold for the elementary transvections.

If group G acts on group H from the left, we will denote the image of $h \in H$ under the homomorphism corresponding to the element $g \in G$ by gh , the element ${}^gh \cdot h^{-1}$ by $\llbracket g, h \rrbracket$ and the element $h \cdot {}^gh^{-1}$ by $[h, g]$.

Definition. Define the *relative symplectic Steinberg group* $\text{StSp}_{2l}(R, I, \Gamma)$ corresponding to the form ideal (I, Γ) as a formal group with the action of the absolute Steinberg group $\text{StSp}_{2l}(R)$ defined by the set of (relative) generators $\{Y_{ij}(a) \mid i \notin \{\pm j\}, a \in I\} \cup \{Y_{i,-i}(\alpha) \mid \alpha \in \Gamma\}$ subject to the following relations

$$Y_{ij}(a) = Y_{-j,-i}(-a\varepsilon_i\varepsilon_j), \quad (\text{KL0})$$

$$Y_{ij}(a)Y_{ij}(b) = Y_{ij}(a+b), \quad (\text{KL1})$$

$$[X_{ij}(r), Y_{hk}(a)] = 1, \text{ for } h \notin \{j, -i\}, k \notin \{i, -j\}, \quad (\text{KL2})$$

$$[X_{ij}(r), Y_{jk}(a)] = Y_{ik}(ra), \text{ for } i \notin \{-j, -k\}, j \neq -k, \quad (\text{KL3})$$

$$[X_{i,-i}(r), Y_{-i,j}(a)] = Y_{ij}(ra\varepsilon_i)Y_{-j,j}(-ra^2), \quad (\text{KL4})$$

$$[Y_{i,-i}(\alpha), X_{-i,j}(r)] = Y_{ij}(\alpha r\varepsilon_i)Y_{-j,j}(-\alpha r^2), \quad (\text{KL5})$$

$$[X_{ij}(r), Y_{j,-i}(a)] = X_{i,-i}(2ra\varepsilon_i), \quad (\text{KL6})$$

$$X_{ij}(a)Y_{hk}(b) = Y_{ij}(a)Y_{hk}(b). \quad (\text{KL7})$$

In other words we consider a free group generated by symbols $(g, x) = {}^g x$ where g is from the absolute Steinberg group and x is from the set of relative generators, $\text{StSp}_{2l}(R)$ naturally acts on this free group via ${}^f(g, x) = (fg, x)$ and then we define a relative symplectic Steinberg group as a factor of the described free group modulo normal subgroup generated by KL0–KL7.

Definition. Obviously, there is a natural map

$$\varphi : \text{StSp}_{2l}(R, I, \Gamma) \rightarrow \text{Sp}_{2l}(R).$$

Then its kernel is denoted by $\text{K}_2\text{Sp}_{2l}(R, I, \Gamma)$.

2 Auxiliary constructions

Definition. Define the *relative Steinberg unipotent radical*

$${}^{(i)}U_1 = \langle Y_{ij}(a), Y_{i,-i}(\alpha) \mid i \notin \{\pm j\}, a \in I, \alpha \in \Gamma \rangle \leq \text{StSp}_{2l}(R, I, \Gamma)$$

and the (*absolute*) *Steinberg parabolic subgroup*

$${}^{(i)}P_1 = \langle X_{kh}(a) \mid i \notin \{h, -k\}, a \in R \rangle \leq \text{StSp}_{2l}(R).$$

Lemma 4 (Levi decomposition). *For $g \in {}^{(i)}P_1$, $u \in {}^{(i)}U_1$ one has*

$${}^g u \in {}^{(i)}U_1.$$

Lemma 5. *One has*

$$[{}^{(i)}U_1, {}^{(i)}U_1] \leq \langle Y_{i,-i}(\alpha) \rangle, \quad [{}^{(i)}U_1, \langle Y_{i,-i}(\alpha) \rangle] = 1.$$

Corollary. *Every element of ${}^{(i)}U_1$ can be expressed in the form*

$$Y_{i,-i}(\alpha)Y_{i,-l}(a_{-l}) \dots Y_{i,-1}(a_{-1})Y_{i,1}(a_1) \dots Y_{i,l}(a_l).$$

Lemma 6. *The restriction of the natural projection $\varphi: \text{StSp}_{2l}(R, I, \Gamma) \twoheadrightarrow \text{Sp}_{2l}(R)$ to ${}^{(i)}U_1$ is injective*

$${}^{(i)}U_1 \cong \varphi({}^{(i)}U_1).$$

Proof. Take an element $x \in {}^{(i)}U_1$. Using the above corollary, decompose x as

$$x = Y_{i,-i}(a_{-i})Y_{i,-l}(a_{-l}) \dots Y_{i,-1}(a_{-1})Y_{i,1}(a_1) \dots Y_{i,l}(a_l).$$

Then $\varphi(x) = 1$ implies that $a_i = 0$ for all i . □

Lemma 7. *Take $v \in I^{2l}$ such that $v_{-i} = 0$ and $\alpha \in \Gamma$. Denote*

$$v_- = \sum_{k < 0} e_k v_k \quad \text{and} \quad v_+ = \sum_{k > 0} e_k v_k.$$

Then

$$\begin{aligned} T(e_i, v, \langle v_-, v_+ \rangle + \alpha) &= T_{i,-i}(\alpha + 2v_i) \cdot \\ &\quad \cdot T_{-l,-i}(v_{-l}\varepsilon_i) \dots T_{-1,-i}(v_{-1}\varepsilon_i)T_{1,-i}(v_1\varepsilon_i) \dots T_{l,-i}(v_l\varepsilon_i). \end{aligned}$$

Proof. Assume that $i > 0$, for $i < 0$ the proof looks exactly the same. Since $T_{j,-i}(v_j\varepsilon_i) = T(e_i, e_j v_j, 0)$ for $j \neq -i$ and

$$T_{i,-i}(2v_i) = T(e_i, 0, 2v_i) = T(e_i, e_i v_i, 0),$$

one has

$$\begin{aligned} T_{-l,-i}(v_{-l}\varepsilon_i) \dots T_{-1,-i}(v_{-1}\varepsilon_i) &= \\ &= T(e_i, e_{-l}v_{-l}, 0) \dots T(e_i, e_{-1}v_{-1}, 0) = T(e_i, v_-, 0), \end{aligned}$$

and

$$\begin{aligned} T_{i,-i}(2v_i)T_{1,-i}(v_1\varepsilon_i) \dots T_{l,-i}(v_l\varepsilon_i) &= \\ &= T(e_i, e_i v_i, 0)T(e_i, e_1 v_1, 0) \dots T(e_i, e_l v_l, 0) = T(e_i, v_+, 0), \end{aligned}$$

so that the right hand side of the desired equality is in fact equal to

$$\begin{aligned} T_{i,-i}(\alpha)T(e_i, v_-, 0)T(e_i, v_+, 0) &= \\ &= T(e_i, 0, \alpha)T(e_i, v, \langle v_-, v_+ \rangle) = T(e_i, v, \langle v_-, v_+ \rangle + \alpha). \end{aligned}$$

□

Definition. For $v \in I^{2l}$ with $v_{-i} = 0$ and $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, define

$$Y(e_i, v, a) = (\phi|_{(i)U_1})^{-1}(T(e_i, v, a)).$$

Remark. By Lemma 7, $T(e_i, v, a)$ indeed lies in $\phi^{(i)}U_1$. Moreover, the same lemma provides the following decomposition.

Lemma 8. For $v \in I^{2l}$ such that $v_{-i} = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$ one has

$$\begin{aligned} Y(e_i, v, a) &= Y_{i,-i}(a + 2v_i - \langle v_-, v_+ \rangle) \cdot \\ &\quad \cdot Y_{-l,-i}(v_{-l}\varepsilon_i) \dots Y_{-1,-i}(v_{-1}\varepsilon_i)Y_{1,-i}(v_1\varepsilon_i) \dots Y_{l,-i}(v_l\varepsilon_i). \end{aligned}$$

Corollary. In particular, $Y(e_{-j}, -e_i a \varepsilon_j, 0) = Y_{ij}(a)$ for $i \notin \{\pm j\}$ and $Y(e_i, 0, \alpha) = Y_{i,-i}(\alpha)$.

Lemma 9. For $v, w \in I^{2l}$ such that $v_{-i} = w_{-i} = 0$ and $a, b \in R$ such that $a - \langle v_-, v_+ \rangle, b - \langle w_-, w_+ \rangle \in \Gamma$, one has

$$Y(e_i, v, a)Y(e_i, w, b) = Y(e_i, v + w, a + b + \langle v, w \rangle).$$

Proof. Obviously, $(v + w)_{-i} = 0$. Moreover,

$$a + b + \langle v, w \rangle - \langle (v + w)_-, (v + w)_+ \rangle = a - \langle v_-, v_+ \rangle + b - \langle w_-, w_+ \rangle \in \Gamma,$$

so that the right hand side of this equality is well-defined. Now, it remains to observe that the images of the elements on both sides under φ coincide. □

Corollary. One has $Y(e_i, 0, 0) = 1$ and $Y(e_i, v, a)^{-1} = Y(e_i, -v, -a)$.

Lemma 10. For $f \in \text{Ep}_{2l}(R)$, $v \in I^{2l}$ one has

$$\langle (fv)_-, (fv)_+ \rangle - \langle v_-, v_+ \rangle \in \Gamma.$$

Proof. We may assume that f is an elementary transvection. For short root transvection one has

$$\begin{aligned} \langle (T_{ij}(r)v)_-, (T_{ij}(r)v)_+ \rangle - \langle v_-, v_+ \rangle &= \\ &= -(v_i + v_j r)v_{-i} - v_j(v_{-j} - v_{-i}r\varepsilon_i\varepsilon_j) + v_i v_{-i} + v_j v_{-j} = \\ &= v_{-i}v_j r(\varepsilon_i\varepsilon_j - 1) \in 2I \subseteq \Gamma. \end{aligned}$$

For long root transvection one has

$$\begin{aligned} \langle (T_{i,-i}(r)v)_-, (T_{i,-i}(r)v)_+ \rangle - \langle v_-, v_+ \rangle &= \\ &= -(v_i + v_{-i}r\varepsilon_i)v_{-i} + v_i v_{-i} = -\varepsilon_i r v_{-i}^2 \in \Gamma. \end{aligned}$$

□

Lemma 11. For $g \in {}^{(i)}P_1$, $v \in I^{2l}$ such that $v_{-i} = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, one has

$${}^g Y(e_i, v, a) = Y(e_i, \phi(g)v, a).$$

Proof. First, observe that since $T_{kh}(a)e_i = e_i$ for $i \notin \{h, -k\}$ one has $\phi(g)e_i = e_i$. Thus,

$$\langle e_i, \phi(g)v \rangle = \langle \phi(g)e_i, \phi(g)v \rangle = \langle e_i, v \rangle = 0,$$

i.e., $(\phi(g)v)_{-i} = 0$. By Lemma 10, $a - \langle (\phi(g)v)_-, (\phi(g)v)_+ \rangle \in \Gamma$, so the right hand side of the desired equation is well-defined. Finally, observe that the images of both sides under φ coincide. □

Lemma 12. For $j \neq -i$, $a \in I$, one has $Y(e_i, e_j a, 0) = Y(e_j, e_i a, 0)$.

Proof. For $i = j$ the claim is obvious. Let $i \neq j$, then

$$Y(e_i, e_j a, 0) = Y_{-j,i}(a\varepsilon_i) = Y_{-i,j}(-a\varepsilon_j) = Y(e_j, e_i a, 0).$$

□

Remark. In the absolute situation $(I, \Gamma) = (R, R)$, we will write $X(e_i, v, a)$ instead of $Y(e_i, v, a)$. These are exactly the elements, which appear in the another presentation.

Lemma 13. For $v \in V$ such that $v_{-j} = v_k = v_{-k} = 0$, $k \notin \{\pm j\}$, $a \in R$, $b \in I$ one has

$$[Y(e_k, e_j b, 0), X(e_{-k}, v, a)] = Y(e_j, vb\varepsilon_k, ab^2)Y(e_{-k}, -e_j ab\varepsilon_{-k}, 0).$$

Proof. The proof of Lemma 12 from the Another presentation paper can be repeated verbatim. \square

Corollary. For $v \in V$ such that $v_{-j} = v_k = v_{-k} = 0$, $k \notin \{\pm j\}$, $a \in R$, $b \in I$ one has the following decomposition

$$Y(e_j, vb, ab^2) = [Y(e_k, e_j b, 0), X(e_{-k}, v\varepsilon_k, a)]Y(e_{-k}, e_j ab\varepsilon_{-k}, 0).$$

Lemma 14. For $j \notin \{\pm k\}$, $v \in I^{2l}$ such that $v_{-j} = v_k = v_{-k} = 0$, $r \in R$, a such that $a - \langle v_-, v_+ \rangle \in \Gamma$, one has

$$[X(e_k, e_j r, 0), Y(e_{-k}, v, a)] = Y(e_j, vr\varepsilon_k, ar^2)Y(e_{-k}, e_j ar\varepsilon_k, 0).$$

Proof. Denote $x = X_{j,-k}(r\varepsilon_k) = X(e_k, e_j r, 0)$, $w = Y_{-k,k}(a - \langle v_-, v_+ \rangle)$, $y = Y_{-l,k}(v_{-l}\varepsilon_{-k}) \cdots Y_{-1,k}(v_{-1}\varepsilon_{-k})$, $z = Y_{1,k}(v_1\varepsilon_{-k}) \cdots Y_{l,k}(v_l\varepsilon_{-k})$. Then

$$[X(e_k, e_j r, 0), Y(e_{-k}, v, a)] = [x, yzw] = [x, y] \cdot {}^y[x, z] \cdot {}^{yz}[x, w].$$

Assume that $j < 0$, for $j > 0$ the proof looks exactly the same. For $h \notin \{\pm j, \pm k\}$ one has

$$\begin{aligned} [X_{j,-k}(r\varepsilon_k), Y_{h,k}(v_h\varepsilon_{-k})] &= [X_{j,-k}(r\varepsilon_k), Y_{-k,-h}(v_h\varepsilon_h)] = \\ &= Y_{j,-h}(rv_h\varepsilon_k\varepsilon_h) = Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j). \end{aligned}$$

For $h = j$ one has

$$[X_{j,-k}(r\varepsilon_k), Y_{j,k}(v_j\varepsilon_{-k})] = [X_{j,-k}(r\varepsilon_k), Y_{-k,-j}(v_j\varepsilon_j)] = Y_{j,-j}(2rv_j\varepsilon_k).$$

Since any $Y_{h,-j}(\hat{a})$ commutes with any $Y_{tk}(\hat{b})$ for $h, t \notin \{\pm k, -j\}$, $h, t < 0$, we have

$$\begin{aligned} [x, y] &= \prod_{h < 0} [X_{j,-k}(r\varepsilon_k), Y_{h,k}(v_h\varepsilon_{-k})] = \\ &= Y_{j,-j}(2rv_j\varepsilon_k) \prod_{\substack{h \neq j \\ h < 0}} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j) = Y(e_j, v_- r\varepsilon_k, 0). \end{aligned}$$

Similarly,

$$[x, z] = \prod_{h > 0} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j) = Y(e_j, v_+ r\varepsilon_k, 0).$$

Then,

$$\begin{aligned}
{}^y\llbracket x, z \rrbracket &= Y(e_{-k}, v_-, 0)Y(e_j, v_+r\varepsilon_k, 0) = \\
&= Y(e_j, v_+r\varepsilon_k, 0)[Y(e_j, -v_+r\varepsilon_k, 0), Y(e_{-k}, v_-, 0)] = \\
&= Y(e_j, v_+r\varepsilon_k, 0)\llbracket X(e_j, -v_+r\varepsilon_k, 0), Y(e_{-k}, v_-, 0) \rrbracket = \\
&= Y(e_j, v_+r\varepsilon_k, 0)Y(e_{-k}, e_j\langle v_-, v_+ \rangle r\varepsilon_k, 0).
\end{aligned}$$

Next,

$$\begin{aligned}
\llbracket x, w \rrbracket &= \llbracket X_{j,-k}(r\varepsilon_k), Y_{-k,k}(a - \langle v_-, v_+ \rangle) \rrbracket = \\
&= \left([Y_{-k,k}(a - \langle v_-, v_+ \rangle), X_{k,-j}(r\varepsilon_j)] \right)^{-1} = \\
&= \left(Y_{-k,-j}((a - \langle v_-, v_+ \rangle)r\varepsilon_j\varepsilon_{-k})Y_{j,-j}(-(a - \langle v_-, v_+ \rangle)r^2) \right)^{-1} = \\
&= Y_{j,-j}(ar^2 - \langle v_-r\varepsilon_k, v_+r\varepsilon_k \rangle)Y(e_j, e_{-k}(a - \langle v_-, v_+ \rangle)r\varepsilon_k, 0).
\end{aligned}$$

Obviously, $\llbracket x, w \rrbracket$ commutes with y and z , thus, finally,

$$\begin{aligned}
\llbracket x, yzw \rrbracket &= Y_{j,-j}(ar^2 - \langle v_-r\varepsilon_k, v_+r\varepsilon_k \rangle)Y_{j,-j}(2rv_j\varepsilon_k) \cdot \\
&\cdot \prod_{\substack{h \neq j \\ h < 0}} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j) \prod_{h > 0} Y_{h,-j}(rv_h\varepsilon_k\varepsilon_j)Y(e_{-k}, e_j\langle v_-, v_+ \rangle r\varepsilon_k, 0) \cdot \\
&\cdot Y(e_{-k}, e_j(a - \langle v_-, v_+ \rangle)r\varepsilon_k, 0) = Y(e_j, vr\varepsilon_k, ar^2)Y(e_{-k}, e_jar\varepsilon_k, 0).
\end{aligned}$$

□

Corollary. For $j \notin \{\pm k\}$, $v \in I^{2l}$ such that $v_{-j} = v_k = v_{-k} = 0$, $r \in R$, a such that $a - \langle v_-, v_+ \rangle \in \Gamma$, one has

$$Y(e_j, vr, ar^2) = \llbracket X(e_k, e_jr, 0), Y(e_{-k}, v\varepsilon_k, a) \rrbracket Y(e_{-k}, e_jar\varepsilon_{-k}, 0).$$

Definition. For $u \in V$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, $u_i = u_{-i} = v_i = v_{-i} = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$ denote

$$Y_{(i)}(u, v, a) = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a) \rrbracket Y(e_{-i}, ua\varepsilon_{-i}, 0).$$

Remark. Due to Lemma 2 one has $\varphi(Y_{(i)}(u, v, a)) = T(u, v, a)$.

Remark. For $v \in I^{2l}$ such that $v_{-j} = v_i = v_{-i} = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, one has

$$Y_{(i)}(e_j, v, a) = Y(e_j, v, a)$$

by Lemma 14. One can also obtain the following result.

Lemma 15. For $v \in V$ with $v_{-j} = v_i = v_{-i} = 0$, $b \in I$ one has

$$Y_{(i)}(v, e_j b, 0) = Y(e_j, vb, 0).$$

Proof. The proof of Lemma 15 from the Another presentation paper can be repeated verbatim. \square

Lemma 16. Consider $u \in V$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, $u_i = u_{-i} = u_j = u_{-j} = 0$, $v_i = v_{-i} = v_j = v_{-j} = 0$, $r \in R$ and $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$. Then one has

$$Y_{(i)}(u, vr, ar^2) = Y_{(j)}(ur, v, a).$$

Proof. Denote $x = X(e_i, u, 0)$, $q = Y(e_{-i}, vr\varepsilon_i, ar^2)$, $c = Y(e_{-i}, uar^2\varepsilon_{-i}, 0)$, then by the very definition

$$Y_{(i)}(u, vr, ar^2) = \llbracket x, q \rrbracket c.$$

Denote $y = X(e_j, e_{-i}r\varepsilon_{-i}, 0)$, $z = Y(e_{-j}, v\varepsilon_j, a)$, $w = Y(e_{-j}, e_{-i}ar\varepsilon_i\varepsilon_{-j}, 0)$, then by Lemma 14

$$q = \llbracket y^{-1}, z \rrbracket w$$

and

$$\llbracket x, q \rrbracket = \llbracket x, \llbracket y^{-1}, z \rrbracket w \rrbracket = \llbracket x, \llbracket y^{-1}, z \rrbracket \rrbracket \cdot \llbracket y^{-1}, z \rrbracket \llbracket x, w \rrbracket.$$

Using Hall–Witt identity,

$$\llbracket x, \llbracket y^{-1}, z \rrbracket \rrbracket = y^{-1}x \llbracket [x^{-1}, y], z \rrbracket \cdot y^{-1}z \llbracket [z^{-1}, x], y \rrbracket.$$

One can check using Lemma 11 that x acts trivially on z , i.e., $[z^{-1}, x] = 1$, thus,

$$\llbracket x, \llbracket y^{-1}, z \rrbracket \rrbracket = y^{-1}x \llbracket [x^{-1}, y], z \rrbracket.$$

Next,

$$[x^{-1}, y] = [X(e_i, -u, 0), X(e_j, -e_{-i}r\varepsilon_i, 0)] = X(-u, -e_jr, 0) = X(e_j, ur, 0).$$

Denote $d = [x^{-1}, y] = X(e_j, ur, 0)$, one can show that $[x, d] = 1$, thus,

$$\llbracket x, \llbracket y^{-1}, z \rrbracket \rrbracket = y^{-1} \llbracket d, z \rrbracket.$$

Observe that y acts trivially on q , thus, ${}^y \llbracket x, q \rrbracket = \llbracket yx, q \rrbracket$. Since $yx = xy[y^{-1}, x^{-1}]$ and

$$[y^{-1}, x^{-1}] = [X(e_{-i}, e_jr\varepsilon_i, 0), X(e_i, -u, 0)] = X(e_j, ur, 0)$$

also acts trivially on q , one has ${}^y\llbracket x, q \rrbracket = \llbracket x, q \rrbracket$, thus,

$$\llbracket x, q \rrbracket = {}^y\llbracket x, q \rrbracket = {}^y\llbracket x, \llbracket y^{-1}, z \rrbracket \rrbracket \cdot {}^y\llbracket y^{-1}, z \rrbracket \llbracket x, w \rrbracket = \llbracket d, z \rrbracket \cdot {}^y\llbracket y^{-1}, z \rrbracket \llbracket x, w \rrbracket.$$

Denote

$$h = \llbracket x, w \rrbracket = \llbracket X(e_i, u, 0), Y(e_{-i}, e_{-j}ar\varepsilon_i\varepsilon_{-j}, 0) \rrbracket = Y(e_{-j}, uar\varepsilon_{-j}, 0),$$

then $\llbracket w, h \rrbracket = 1$, thus, $\llbracket y^{-1}, z \rrbracket h = \llbracket y^{-1}, z \rrbracket w h = {}^q h = h$, so,

$$\llbracket x, q \rrbracket = \llbracket d, z \rrbracket \cdot {}^y h = \llbracket d, z \rrbracket \cdot h \cdot [h^{-1}, y].$$

Since

$$\llbracket y, h^{-1} \rrbracket = \llbracket X(e_j, -e_{-i}r\varepsilon_i, 0), Y(e_{-j}, uar\varepsilon_j, 0) \rrbracket = Y(e_{-i}, uar^2\varepsilon_{-i}, 0) = c,$$

one has

$$\llbracket x, q \rrbracket c = \llbracket d, z \rrbracket h$$

or,

$$Y_{(i)}(u, vr, ar^2) = Y_{(j)}(ur, v, a).$$

□

Definition. Define the (absolute) Levi subgroup ${}^{(i)}L_1 = {}^{(i)}P_1 \cap {}^{(-i)}P_1$.

Remark. Observe that for $g \in {}^{(i)}L_1$ one has $\phi(g)e_i = e_i$ and $\phi(g)e_{-i} = e_{-i}$. Indeed, the first equality holds for $g = X_{kh}(a)$ with $\{-k, h\} \not\ni i$ and the second one for $g = X_{kh}(a)$ with $\{-k, h\} \not\ni -i$.

Lemma 17. For $u \in V$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, $u_i = u_{-i} = v_i = v_{-i} = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, $g \in {}^{(i)}L_1$, one has

$$gY_{(i)}(u, v, a)g^{-1} = Y_{(i)}(\phi(g)u, \phi(g)v, a).$$

Remark. Since $\langle \phi(g)u, e_i \rangle = \langle \phi(g)u, \phi(g)e_i \rangle = \langle u, e_i \rangle = 0$, one can see that $(\phi(g)u)_{-i} = 0$ and similarly $(\phi(g)u)_i = (\phi(g)v)_{-i} = (\phi(g)v)_i = 0$. Using Lemma 10 one obtains, that $a - \langle (\phi(g)v)_-, (\phi(g)v)_+ \rangle \in \Gamma$. Thus, $Y_{(i)}(\phi(g)u, \phi(g)v, a)$ is well-defined.

Proof. The proof of Lemma 17 from the Another presentation paper works for this situation as well. □

Remark. Lemma 7 implies that for v such that $v_{-i} = v_j = v_{-j} = 0$, $j \notin \{\pm i\}$ one has $Y(e_i, v, a) \in {}^{(j)}L_1$.

Remark. For w orthogonal to both u and v , $\langle u, w \rangle = \langle v, w \rangle = 0$, one has $T(u, v, a)w = w$. Below, in the computations this fact is frequently used without any special reference.

Lemma 18. *For $j \notin \{\pm i\}$ and $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^{2l}$ such that $v_i = v_{-i} = 0$ and $\langle u, v \rangle = 0$, and for $a, b \in I$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, one has*

$$Y_{(i)}(u, v, a)Y_{(i)}(u, e_j b, 0) = Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{-j}).$$

Proof. Start with the right-hand side (one can check that it is well-defined)

$$\begin{aligned} & Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{-j}) = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, (v + e_j b) \varepsilon_i, a + v_{-j} b \varepsilon_{-j}) \rrbracket Y(e_{-i}, u(a + v_{-j} b \varepsilon_{-j}) \varepsilon_{-i}, 0). \end{aligned}$$

Decompose $Y(e_{-i}, (v + e_j b) \varepsilon_i, a + v_{-j} b \varepsilon_{-j})$ inside the commutator and use the familiar identity $\llbracket a, bc \rrbracket = \llbracket a, b \rrbracket \cdot {}^b \llbracket a, c \rrbracket$ to obtain

$$\begin{aligned} & \llbracket X(e_i, u, 0), Y(e_{-i}, (v + e_j b) \varepsilon_i, a + v_{-j} b \varepsilon_{-j}) \rrbracket = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) Y(e_{-i}, e_j b \varepsilon_i, 0) \rrbracket = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket \cdot {}^{Y(e_{-i}, v \varepsilon_i, a)} \llbracket X(e_i, u, 0), Y(e_{-i}, e_j b \varepsilon_i, 0) \rrbracket = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket \cdot {}^{Y(e_{-i}, v \varepsilon_i, a)} Y_{(i)}(u, e_j b, 0) = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket \cdot {}^{Y(e_{-i}, v \varepsilon_i, a)} Y(e_j, ub, 0). \end{aligned}$$

Observe that in general $X(e_{-i}, v \varepsilon_i, a)$ does not lie in ${}^{(j)}P_1$, but $X(e_j, ub, 0)$ always lies in ${}^{(i)}P_1$. So that we can compute the conjugate as follows

$$\begin{aligned} & {}^{Y(e_{-i}, v \varepsilon_i, a)} Y(e_j, ub, 0) = Y(e_j, ub, 0) [Y(e_j, -ub, 0), Y(e_{-i}, v \varepsilon_i, a)] = \\ & = Y(e_j, ub, 0) Y(e_{-i}, T(e_j, -ub, 0) v \varepsilon_i, a) \cdot Y(e_{-i}, -v \varepsilon_i, -a) = \\ & = Y(e_j, ub, 0) Y(e_{-i}, v \varepsilon_i - ub \langle e_j, v \varepsilon_i \rangle, a) Y(e_{-i}, -v \varepsilon_i, -a) = \\ & = Y(e_j, ub, 0) Y(e_{-i}, -ub v_{-j} \varepsilon_i \varepsilon_j, 0). \end{aligned}$$

Thus,

$$\begin{aligned} & Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_{-j}) = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket Y(e_j, ub, 0) \cdot \\ & \cdot Y(e_{-i}, -ub v_{-j} \varepsilon_i \varepsilon_j, 0) Y(e_{-i}, u(a + v_{-j} b \varepsilon_{-j}) \varepsilon_{-i}, 0) = \\ & = \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket Y(e_j, ub, 0) Y(e_{-i}, ua \varepsilon_{-i}, 0). \end{aligned}$$

Finally, it remains to observe that $X(e_j, ub, 0) \in {}^{(-i)}P_1$ and thus one has $[Y(e_j, ub, 0), Y(e_{-i}, ua\varepsilon_{-i}, 0)] = 1$ and

$$\begin{aligned} Y_{(i)}(u, v + e_j b, a + v_{-j} b \varepsilon_j) &= \llbracket X(e_i, u, 0), Y(e_{-i}, v \varepsilon_i, a) \rrbracket Y(e_{-i}, ua\varepsilon_{-i}, 0) \cdot \\ &\quad \cdot Y(e_j, ub, 0) = Y_{(i)}(u, v, a) Y_{(i)}(u, e_j b, 0). \end{aligned}$$

□

Definition. For $u \in V$ such that $u_i = u_{-i} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, and $a \in R$ such that $a - \langle v_{-}, v_{+} \rangle \in \Gamma$ define

$$\begin{aligned} Y_{(i)}(u, v, a) &= \\ &= Y_{(i)}(u, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0). \end{aligned}$$

Remark. Observe that the above definition coincides with the old one for v with $v_i = v_{-i} = 0$, so that we can use the same notation for the generator. Observe also that the right-hand side is well-defined.

Lemma 19. For $g \in {}^{(i)}L_1$, $u \in V$ such that $u_i = u_{-i} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, and $a \in R$ such that $a - \langle v_{-}, v_{+} \rangle \in \Gamma$ one has

$$g Y_{(i)}(u, v, a) g^{-1} = Y_{(i)}(\phi(g)u, \phi(g)v, a).$$

Proof. Since $g \in {}^{(i)}L_1$ one gets

$$(\phi(g)v)_i = \langle \phi(g)v, e_{-i} \rangle \varepsilon_i = \langle \phi(g)v, \phi(g)e_{-i} \rangle \varepsilon_i = \langle v, e_{-i} \rangle \varepsilon_i = v_i$$

and similarly $(\phi(g)v)_{-i} = v_{-i}$. Then

$$\begin{aligned} {}^g Y_{(i)}(u, v, a) &= \\ &= {}^g Y_{(i)}(u, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) \cdot {}^g Y(e_i, uv_i, 0) \cdot {}^g Y(e_{-i}, uv_{-i}, 0) = \\ &= Y_{(i)}(\phi(g)u, \phi(g)v - \phi(g)e_i v_i - \phi(g)e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) \cdot \\ &\quad \cdot Y(e_i, \phi(g)uv_i, 0) Y(e_{-i}, \phi(g)uv_{-i}, 0) = \\ &= Y_{(i)}(\phi(g)u, \phi(g)v - e_i (\phi(g)v)_i - e_{-i} (\phi(g)v)_{-i}, a - (\phi(g)v)_i (\phi(g)v)_{-i} \varepsilon_i) \cdot \\ &\quad \cdot Y(e_i, \phi(g)u (\phi(g)v)_i, 0) Y(e_{-i}, \phi(g)u (\phi(g)v)_{-i}, 0) = \\ &= Y_{(i)}(\phi(g)u, \phi(g)v, a). \end{aligned}$$

□

Remark. Obviously, in the absolute case $(I, \Gamma) = (R, R)$ for $u, v \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $v_j = v_{-j} = 0, j \notin \{\pm i\}, a \in R$, one has that such an element $X_{(i)}(u, v, a)$ lies in ${}^{(j)}L_1$. Indeed, it is a product of elements from ${}^{(j)}L_1$ by definition.

Lemma 20. For $j \notin \{\pm i\}$, $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$, and $a \in R$ such that $a - \langle v_{-}, v_{+} \rangle \in \Gamma$ one has

$$Y_{(i)}(u, v, a) = Y_{(j)}(u, v, a).$$

Proof. Denote $\tilde{v} = v - e_i v_i - e_{-i} v_{-i}$, $\tilde{a} = a - v_i v_{-i} \varepsilon_i$. Then by Lemma 18 one gets

$$\begin{aligned} Y_{(i)}(u, \tilde{v}, \tilde{a}) &= Y_{(i)}(u, \tilde{v} - e_{-j} v_{-j}, \tilde{a} - v_j v_{-j} \varepsilon_j) Y_{(i)}(u, e_{-j} v_{-j}, 0) = \\ &= Y_{(i)}(u, \tilde{v} - e_{-j} v_{-j} - e_j v_j, \tilde{a} - v_j v_{-j} \varepsilon_j) Y_{(i)}(u, e_j v_j, 0) Y_{(i)}(u, e_{-j} v_{-j}, 0). \end{aligned}$$

Further, denote $\tilde{\tilde{v}} = \tilde{v} - e_j v_j - e_{-j} v_{-j}$ and $\tilde{\tilde{a}} = \tilde{a} - v_j v_{-j} \varepsilon_j$. Then one has

$$\begin{aligned} Y_{(i)}(u, v, a) &= Y_{(i)}(u, \tilde{\tilde{v}}, \tilde{\tilde{a}}) Y(e_j, uv_j, 0) Y(e_{-j}, uv_{-j}, 0) \cdot \\ &\quad \cdot Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0). \end{aligned}$$

Changing roles of i and j one gets

$$\begin{aligned} Y_{(j)}(u, v, a) &= Y_{(j)}(u, \tilde{\tilde{v}}, \tilde{\tilde{a}}) Y(e_i, uv_i, 0) Y(e_{-i}, uv_{-i}, 0) \cdot \\ &\quad \cdot Y(e_j, uv_j, 0) Y(e_{-j}, uv_{-j}, 0). \end{aligned}$$

But $Y_{(i)}(u, \tilde{\tilde{v}}, \tilde{\tilde{a}}) = Y_{(j)}(u, \tilde{\tilde{v}}, \tilde{\tilde{a}})$ by Lemma 16. Finally, it remains to observe that $Y(e_{-i}, uv_{-i}, 0)$ and $Y(e_i, uv_i, 0)$ commute with both $Y(e_j, uv_j, 0)$ and $Y(e_{-j}, uv_{-j}, 0)$. This is obvious from the fact that $X(e_{-i}, uv_{-i}, 0)$ and $X(e_i, uv_i, 0)$ lie in ${}^{(j)}L_1$. \square

Remark. For u equal to the base vector e_j using Lemma 13 one gets

$$\begin{aligned} Y_{(i)}(e_j, v, a) &= \\ &= Y_{(i)}(e_j, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_i, e_j v_i, 0) Y(e_{-i}, e_j v_{-i}, 0) = \\ &= Y(e_j, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_j, e_i v_i, 0) Y(e_j, e_{-i} v_{-i}, 0) = \\ &= Y(e_j, v, a). \end{aligned}$$

Definition. For u having at least two pairs of zeros the element $Y_{(i)}(u, v, a)$ does not depend on the choice of i by Lemma 20. In this situation we will often omit the index in the notation,

$$Y(u, v, a) = Y_{(i)}(u, v, a).$$

Definition. For $u \in V$, $w \in I^{2n}$, such that $\langle u, w \rangle = 0$, $w_i = w_{-i} = 0$ and $a \in \langle w_-, w_+ \rangle + \Gamma$ define

$$Z_{(i)}(u, w, a) = Y_{(i)}(u - e_i u_i - e_{-i} u_{-i}, w, a) Y(e_i u_i + e_{-i} u_{-i}, w, a) \cdot \\ \cdot Y(e_i u_i + e_{-i} u_{-i}, (u - e_i u_i - e_{-i} u_{-i})a, 0).$$

Our next objective is to show that $Z_{(i)}(u, 0, a)$ does not depend on the choice of i .

Lemma 21. For $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^{2l}$ such that $v_i = v_{-i} = 0$, $\langle u, v \rangle = 0$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$, $b \in \Gamma$ one has

$$Y(u, v, a + b) = Y(u, v, a) Y(u, 0, b).$$

Proof. Decompose $Y(u, v, a + b)$ as follows

$$Y(u, v, a + b) = Y_{(i)}(u, v, a + b) = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a + b) \rrbracket Y(e_{-i}, u(a + b)\varepsilon_{-i}, 0).$$

Using $\llbracket a, bc \rrbracket = \llbracket a, b \rrbracket \cdot {}^b \llbracket a, c \rrbracket$ we obtain

$$\llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a + b) \rrbracket = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a) Y(e_{-i}, 0, b) \rrbracket = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a) \rrbracket \cdot {}^{Y(e_{-i}, v\varepsilon_i, a)} \llbracket X(e_i, u, 0), Y(e_{-i}, 0, b) \rrbracket.$$

Since $\langle u, v \rangle = 0$ one has

$$X(e_{-i}, v\varepsilon_i, a) Y(e_{-i}, ub\varepsilon_{-i}, 0) = Y(e_{-i}, ub\varepsilon_{-i}, 0),$$

and thus

$$Y(u, v, a + b) = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a + b) \rrbracket Y(e_{-i}, ub\varepsilon_{-i}, 0) Y(e_{-i}, ua\varepsilon_{-i}, 0) = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a) \rrbracket \cdot {}^{Y(e_{-i}, v\varepsilon_i, a)} \llbracket X(e_i, u, 0), Y(e_{-i}, 0, b) \rrbracket \cdot \\ \cdot {}^{X(e_{-i}, v\varepsilon_i, a)} Y(e_{-i}, ub\varepsilon_{-i}, 0) \cdot Y(e_{-i}, ua\varepsilon_{-i}, 0) = \\ = \llbracket X(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, 0) \rrbracket \cdot {}^{Y(e_{-i}, v\varepsilon_i, a)} Y_{(i)}(u, 0, b) \cdot Y(e_{-i}, ua\varepsilon_{-i}, 0).$$

Recall that $X_{(j)}(u, 0, b) \in {}^{(i)}L_1$ acts trivially on both $Y(e_{-i}, v\varepsilon_i, a)$ and $Y(e_{-i}, ua\varepsilon_{-i}, 0)$, so that

$$Y(u, v, a + b) = [Y(e_i, u, 0), Y(e_{-i}, v\varepsilon_i, a)] Y(e_{-i}, ua\varepsilon_{-i}, 0) Y(u, 0, b) = \\ = Y_{(i)}(u, v, a) Y(u, 0, b).$$

□

Remark. For $u \in V$ having at least two pairs of zeros one has $Y(u, 0, 0) = 1$ and $Y(u, 0, a)^{-1} = Y(u, 0, -a)$.

Lemma 22. For $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$, $v \in I^{2l}$ such that $\langle u, v \rangle = 0$ and $v_i v_{-i} \in \Gamma$, $a \in R$ such that $a - \langle v_-, v_+ \rangle \in \Gamma$ one has

$$Y(u, v, a) = Y(u, v - e_i v_i - e_{-i} v_{-i}, a) Y(u, e_i v_i + e_{-i} v_{-i}, 0).$$

Proof. By definition

$$\begin{aligned} Y(u, v, a) &= Y_{(i)}(u, v, a) = \\ &= Y_{(i)}(u, v - e_i v_i - e_{-i} v_{-i}, a - v_i v_{-i} \varepsilon_i) Y(e_i, u v_i, 0) Y(e_{-i}, u v_{-i}, 0). \end{aligned}$$

Denote $\tilde{v} = v - e_i v_i - e_{-i} v_{-i}$, then by the previous lemma

$$Y(u, \tilde{v}, a - v_i v_{-i} \varepsilon_i) = Y(u, \tilde{v}, a) Y(u, 0, -v_i v_{-i} \varepsilon_i),$$

and thus

$$\begin{aligned} Y(u, v, a) &= Y(u, \tilde{v}, a) Y(u, 0, -v_i v_{-i} \varepsilon_i) Y(e_i, u v_i, 0) Y(e_{-i}, u v_{-i}, 0) = \\ &= Y(u, \tilde{v}, a) Y_{(i)}(u, e_i v_i + e_{-i} v_{-i}, 0). \end{aligned}$$

□

Corollary. Consider $u \in V$, $w \in I^{2n}$, such that $\langle u, w \rangle = 0$, $w_i = w_{-i} = 0$, $a \in \langle w_-, w_+ \rangle + \Gamma$, $j \notin \{\pm i\}$ and denote $v = e_i u_i + e_{-i} u_{-i}$, $v' = e_j u_j + e_{-j} u_{-j}$, $\tilde{u} = u - v$, $\tilde{\tilde{u}} = \tilde{u} - v'$. Then one has

$$\begin{aligned} Z_{(i)}(u, w, a) &= Y_{(i)}(\tilde{u}, w, a) Y(v, w, a) Y(v, \tilde{u} a, 0) = \\ &= Y_{(i)}(\tilde{\tilde{u}}, w, a) Y(v, w, a) Y(v, \tilde{\tilde{u}} a, 0) Y(v, v' a, 0). \end{aligned}$$

Proof. Since $l \geq 3$ the vector v has at least two pairs of zeros, so that one can apply the previous lemma. □

Lemma 23. For $j, k \notin \{\pm i\}$, $u, v \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $v_i = v_{-i} = v_k = v_{-k} = 0$, $\langle u, v \rangle = 0$, $w \in I^{2l}$ such that $w_i = w_{-i} = 0$, $\langle u, w \rangle = \langle v, w \rangle = 0$ and $a \in \langle w_-, w_+ \rangle + \Gamma$ holds

$$Y_{(i)}(u + v, w, a) = Y(u, w, a) \cdot Y(v, w, a) \cdot Y(v, u a, 0).$$

Proof. Decomposing $X(e_i, u+v, 0) = X(e_i, v, 0)X(e_i, u, 0)$ inside the commutator and using $\llbracket ab, c \rrbracket = {}^a\llbracket b, c \rrbracket \cdot \llbracket a, c \rrbracket$ we get

$$\begin{aligned} Y_{(i)}(u+v, w, a) &= \llbracket X(e_i, u+v, 0), Y(e_{-i}, w, a) \rrbracket Y(e_{-i}, (u+v)a\varepsilon_{-i}, 0) = \\ &= \llbracket X(e_i, v, 0)X(e_i, u, 0), Y(e_{-i}, w, a) \rrbracket Y(e_{-i}, va\varepsilon_{-i}, 0)Y(e_{-i}, ua\varepsilon_{-i}, 0) = \\ &= {}^{X(e_i, v, 0)}\llbracket X(e_i, u, 0), Y(e_{-i}, w, a) \rrbracket \cdot \llbracket X(e_i, v, 0), Y(e_{-i}, w, a) \rrbracket \cdot \\ &\cdot Y(e_{-i}, va\varepsilon_{-i}, 0)Y(e_{-i}, ua\varepsilon_{-i}, 0) = {}^{X(e_i, v, 0)}\llbracket X(e_i, u, 0), Y(e_{-i}, w, a) \rrbracket \cdot \\ &\cdot Y_{(i)}(v, w, a)Y(e_{-i}, ua\varepsilon_{-i}, 0). \end{aligned}$$

Observe that $Y_{(k)}(v, w, a) \in {}^{(i)}L_1$ commutes with $Y(e_{-i}, -ua\varepsilon_{-i}, 0)$, moreover, $X(e_i, v, 0) \in {}^{(k)}L_1$ and acts trivially on $Y_{(k)}(v, w, a)$, thus $Y_{(k)}(v, w, a)$ commutes with ${}^{X(e_i, v, 0)}Y(e_{-i}, -ua\varepsilon_{-i}, 0)$. Thus, we get

$$\begin{aligned} {}^{X(e_i, v, 0)}\llbracket X(e_i, u, 0), Y(e_{-i}, w, a) \rrbracket \cdot Y(v, w, a) \cdot Y(e_{-i}, ua\varepsilon_{-i}, 0) &= \\ = {}^{X(e_i, v, 0)}\llbracket X(e_i, u, 0), Y(e_{-i}, w, a) \rrbracket \cdot {}^{X(e_i, v, 0)}Y(e_{-i}, ua\varepsilon_{-i}, 0) \cdot \\ \cdot {}^{X(e_i, v, 0)}Y(e_{-i}, -ua\varepsilon_{-i}, 0) \cdot Y(v, w, a) \cdot Y(e_{-i}, ua\varepsilon_{-i}, 0) &= \\ = {}^{X(e_i, v, 0)}Y_{(i)}(u, w, a) \cdot Y(v, w, a) \cdot \llbracket X(e_i, v, 0), Y(e_{-i}, -ua\varepsilon_{-i}, 0) \rrbracket &= \\ = {}^{X(e_i, v, 0)}Y(u, w, a) \cdot Y(v, w, a) \cdot Y(v, ua, 0). \end{aligned}$$

It only remains to show that

$${}^{X(e_i, v, 0)}Y(u, w, a) = Y(u, w, a).$$

Decompose

$$Y_{(-i)}(u, w, a) = \llbracket X(e_{-i}, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(e_i, ua\varepsilon_i, 0),$$

then,

$$\begin{aligned} {}^{X(e_i, v, 0)}Y(u, w, a) &= \\ = {}^{X(e_i, v, 0)}\llbracket X(e_{-i}, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(e_i, ua\varepsilon_i, 0) &= \\ = \llbracket X(e_{-i} + v\varepsilon_i, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(e_i, ua\varepsilon_i, 0) &= \\ = \llbracket X(e_{-i}, u, 0)X(v\varepsilon_i, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(e_i, ua\varepsilon_i, 0) &= \\ = {}^{X(e_{-i}, u, 0)}\llbracket X(v\varepsilon_i, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket \cdot \\ \cdot \llbracket X(e_{-i}, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(e_i, ua\varepsilon_i, 0) &= \\ = {}^{X(e_{-i}, u, 0)}\llbracket X(v\varepsilon_i, u, 0), Y(e_i, w\varepsilon_{-i}, a) \rrbracket Y(u, w, a). \end{aligned}$$

Finally, notice that $X_{(k)}(v\varepsilon_i, u, 0) \in {}^{(i)}L_1$ commutes with $Y(e_i, w\varepsilon_{-i}, a)$. \square

3 Case of maximal form parameter

In this section we assume that $\Gamma = I$.

Lemma 24. *For $k, j \notin \{\pm i\}$, $u, v \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $v_i = v_{-i} = v_k = v_{-k} = 0$, $\langle u, v \rangle = 0$, and $a \in I$, one has*

$$Y(u, va, 0) = Y(v, ua, 0).$$

Proof. By Lemma 23 one has

$$Y(v, ua, 0) = Y(v, 0, -a)Y(u, 0, -a)Y_{(i)}(u + v, 0, a)$$

and similarly

$$Y(u, va, 0) = Y(u, 0, -a)Y(v, 0, -a)Y_{(i)}(v + u, 0, a).$$

But $X_{(j)}(u, 0, -a) \in {}^{(i)}L_1$ acts trivially on $Y_{(i)}(v, 0, -a)$. \square

Lemma 25. *For $u \in V$, $a \in I$ and $j \notin \{\pm i\}$*

$$Z_{(i)}(u, 0, a) = Z_{(j)}(u, 0, a).$$

Proof. Set

$$v = e_i u_i + e_{-i} u_{-i}, \quad v' = e_j u_j + e_{-j} u_{-j}, \quad \tilde{u} = u - v, \quad \tilde{\tilde{u}} = \tilde{u} - v'.$$

Lemmas 22 and 23 imply that

$$\begin{aligned} Z_{(i)}(u, 0, a) &= Y_{(i)}(\tilde{u}, 0, a)Y(v, 0, a)Y(v, \tilde{u}a, 0) = \\ &= Y_{(i)}(\tilde{\tilde{u}}, 0, a)Y(v, 0, a)Y(v, \tilde{\tilde{u}}a, 0)Y(v, v'a, 0) = \\ &= Y(\tilde{\tilde{u}}, 0, a)Y(v', 0, a)Y(v', \tilde{\tilde{u}}a, 0)Y(v, 0, a)Y(v, \tilde{\tilde{u}}a, 0)Y(v, v'a, 0). \end{aligned}$$

Interchanging roles of i and j one has

$$\begin{aligned} Z_{(j)}(u, 0, a) &= \\ &= Y(\tilde{\tilde{u}}, 0, a)Y(v, 0, a)Y(v, \tilde{\tilde{u}}a, 0)Y(v', 0, a)Y(v', \tilde{\tilde{u}}a, 0)Y(v', va, 0). \end{aligned}$$

By Lemma 24 one has $Y(v, v'a, 0) = Y(v', va, 0)$. Now we have to show that $Y(v, 0, a)Y(v, \tilde{\tilde{u}}a, 0)$ commutes with $Y(v', 0, a)Y(v', \tilde{\tilde{u}}a, 0)$. With this end fix $k \notin \{\pm i, \pm j\}$. Observe that $Y_{(i)}(v', 0, a) \in {}^{(k)}L_1$ commutes with $Y_{(k)}(v, \tilde{\tilde{u}}a, 0)$ by Lemma 19. Next, we will show that

$$[Y(v, \tilde{\tilde{u}}a, 0), Y(v', \tilde{\tilde{u}}a, 0)] = 1.$$

By Lemma 24 one has

$$Y(v, \tilde{u}a, 0) = Y(\tilde{u}, va, 0)$$

and

$$Y(v', \tilde{u}a, 0) = Y(\tilde{u}, v'a, 0).$$

Now, Lemma 22 implies that

$$\begin{aligned} Y(v, \tilde{u}a, 0)Y(v', \tilde{u}a, 0) &= Y(\tilde{u}, va, 0)Y(\tilde{u}, v'a, 0) = \\ &= Y(\tilde{u}, va + v'a, 0) = Y(\tilde{u}, v'a, 0)Y(\tilde{u}, va, 0) = \\ &= Y(v', \tilde{u}a, 0)Y(v, \tilde{u}a, 0). \end{aligned}$$

Finally, it remains to observe that $Y_{(j)}(v, 0, a) \in {}^{(k)}L_1$ commutes with both $Y_{(k)}(v', 0, a)$ and $Y_{(k)}(v', \tilde{u}a, 0)$ by Lemma 19. \square

Remark. Since $Z_{(i)}(u, 0, a)$ does not depend on the choice of i we will often omit the index in the notation

$$Z(u, 0, a) = Z_{(i)}(u, 0, a).$$

Our next objective is to prove the following formula describing the action of $\text{StSp}_{2l}(R)$ on the long-root type generators, namely

$${}^gZ(u, 0, a) = Z(\phi(g)u, 0, a).$$

Lemma 26. *For any $u \in V$, any $a \in \Gamma$, $b \in R$ and any index i , one has*

$$X_{i,-i}(b)Z(u, 0, a) = Z(T_{i,-i}(b)u, 0, a).$$

Proof. The proof of Lemma 26 from the Another Presentation paper can be repeated verbatim. \square

Lemma 27. *For $u \in V$, $v, w \in I^{2l}$ and $j \notin \{\pm i\}$ such that*

$$u_i = u_{-i} = u_j = u_{-j} = 0, \quad v_j = v_{-j} = 0, \quad w_j = w_{-j} = 0,$$

and $\langle u, v \rangle = 0$, $\langle u, w \rangle = 0$, $\langle v, w \rangle = 0$, one has

$$Y(u, v, 0)Y(u, w, 0) = Y(u, v + w, 0).$$

Proof. The proof of Lemma 27 from the Another Presentation paper can be repeated verbatim. \square

Lemma 28. Let $\text{Card}\{\pm i, \pm j, \pm k\} = 6$ and let $v \in I^{2l}$, $v' \in V$ be vectors having only $\pm i$ -th and $\pm j$ -th non-zero coordinates respectively; consider also $v'' \in V$ such that $(v'')_i = (v'')_{-i} = (v'')_j = (v'')_{-j} = 0$. Set $w = v' + v''$. Then

$$Y(v'', v, 0)Y(v', v, 0) = Y_{(i)}(w, v, 0).$$

Proof. The proof of Lemma 28 from the Another Presentation paper can be repeated verbatim. \square

Lemma 29. For any $j \notin \{\pm k\}$, any $u \in V$ and any $a \in I$, $b \in R$, one has

$$X_{jk(b)}Z(u, 0, a) = Z(T_{jk}(b)u, 0, a).$$

Proof. The proof of Lemma 29 from the Another presentation paper can be repeated almost verbatim. The only difference is that one should use another approach to show that $X_{jk(b)}$ acts trivially on $Y(v, 0, a)$ and $Y(v, \tilde{\tilde{u}}a, 0)$. Namely, one should observe first that $X_{jk(b)} \in {}^{(i)}L_1$ acts trivially on

$$Y_{(i)}(\tilde{\tilde{u}}a, v, 0) = Y(v, \tilde{\tilde{u}}a, 0).$$

Afterwards, decompose

$$Y(v, 0, a) = \llbracket X(e_{-j}, v, 0), Y(e_j, 0, a) \rrbracket Y(e_j, va\varepsilon_j, 0),$$

thus,

$$X_{jk(b)}Y(v, 0, a) = \llbracket X(T_{jk}(b)e_{-j}, v, 0), Y(e_j, 0, a) \rrbracket Y(e_j, va\varepsilon_j, 0).$$

Since

$$\begin{aligned} X(T_{jk}(b)e_{-j}, v, 0) &= X(e_{-j} - e_{-k}a\varepsilon_k\varepsilon_j, v, 0) = \\ &= X(e_{-j}, v, 0)X(-e_{-k}a\varepsilon_k\varepsilon_j, v, 0) = X(e_{-j}, v, 0)X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0), \end{aligned}$$

we obtain that

$$\begin{aligned} X_{jk(b)}Y(v, 0, a) &= \\ &= \llbracket X(e_{-j}, v, 0)X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0), Y(e_j, 0, a) \rrbracket Y(e_j, va\varepsilon_j, 0) = \\ &= X^{(e_{-j}, v, 0)} \llbracket X(e_{-k}, -va\varepsilon_k\varepsilon_j, 0), Y(e_j, 0, a) \rrbracket \cdot Y(v, 0, a). \end{aligned}$$

Finally, it remains to observe that $X(e_{-j}, v, 0) \in {}^{(j)}L_1$ acts trivially on $Y(e_j, 0, a)$. \square

Corollary. Lemmas 26 and 29 imply that for any $g \in \text{StSp}(2l, R)$, one has

$${}^gZ(u, 0, a) = Z(\phi(g)u, 0, a).$$

Lemma 30. The set of elements $\{Z(u, 0, a) \mid u \in V, a \in I\}$ generates $\text{StSp}_{2l}(R, I)$ as a group.

Proof. Firstly, choosing some i and j such that $\text{Card}\{\pm i, \pm j\} = 4$ one has

$$Y_{(j)}(e_i, 0, a) = Y(e_i, 0, a)Y(0, 0, a)Y(0, e_i a, 0) = Y(e_i, 0, a) = Z_{i,-i}(a).$$

Now, choosing $k \notin \{\pm i, \pm j\}$, taking $u = e_{-k}$, $v = -e_j \varepsilon_k$ and any $a \in I$, and using Lemma 23 we obtain

$$\begin{aligned} X_{jk}(a) &= Y(u, va, 0) = \\ &= Y(-e_j \varepsilon_k, 0, -a)Y(e_{-k}, 0, -a)Y_{(i)}(e_{-k} - e_j \varepsilon_k, 0, a) = \\ &= Z_{(i)}(-e_j \varepsilon_k, 0, -a)Z_{(i)}(e_{-k}, 0, -a)Z_{(i)}(e_{-k} - e_j \varepsilon_k, 0, a). \end{aligned}$$

□

Corollary. Clearly, $\text{Ker } \phi$ acts trivially on $\text{StSp}_{2l}(R, I)$.

Lemma 31. For $u \in V$, $a, b \in I$ one has

$$Z(u, 0, a)Z(u, 0, b) = Z(u, 0, a + b).$$

Proof. The proof of Lemma 31 from the Another Presentation paper can be repeated verbatim. □

Lemma 32. For $j \notin \{\pm i\}$, $u \in V$ such that $u_i = u_{-i} = u_j = u_{-j} = 0$ and $a \in I$, $b \in R$, one has

$$Y(ub, 0, a) = Y(u, 0, ab^2).$$

Proof. The claim follows directly from Lemma 16. □

Lemma 33. For $u \in V$, $a \in I$, $b \in R$, one has

$$Z(ub, 0, a) = Z(u, 0, ab^2).$$

Proof. The proof of Lemma 33 from the Another Presentation paper can be repeated verbatim. □

Definition. For $u, v \in V$ such that $\langle u, v \rangle = 0$, $a \in I$ set

$$Z(v, u, a, 0) = Z(v, 0, -a)Z(u, 0, -a)Z(u + v, 0, a).$$

Lemma 34. For $g \in \text{StSp}(2l, R)$ and $u, v \in V$ such that $\langle u, v \rangle = 0$, $a \in I$, $b \in R$, one has

- a) $Z(v, u, a, 0) = Z(u, v, a, 0)$;
- b) ${}^gZ(u, v, a, 0) = Z(\phi(g)u, \phi(g)v, a, 0)$;
- c) $Z(u, ub, a, 0) = Z(u, 0, 2ab)$.

Proof. Since a) and b) are obvious, it remains only to check c). By the very definition we have

$$Z(u, ub, a, 0) = Z(u, 0, -a)Z(ub, 0, -a)Z(ub + u, 0, a).$$

Then, using Lemma 33 and then Lemma 31, we get

$$\begin{aligned} Z(u, 0, -a)Z(ub, 0, -a)Z(ub + u, 0, a) &= \\ &= Z(u, 0, -a)Z(u, 0, -ab^2)Z(u, 0, a(b+1)^2) = Z(u, 0, 2ab). \end{aligned}$$

□

Lemma 35. Consider $u, w \in V$ such that

$$\langle u, w \rangle = 0, \quad w_i = w_{-i} = w_j = w_{-j} = 0,$$

where $i \notin \{\pm j\}$. Then

$$Z(u + w, 0, a) = Z(u, 0, a)Z(w, 0, a)Y(w, ua, 0).$$

Proof. The proof of Lemma 35 from the Another Presentation paper can be repeated verbatim. □

Lemma 36. For $v \in V$ such that $v_{-i} = 0$, $a \in I$, one has

$$Z(e_i, v, a, 0) = Y(e_i, va, 0).$$

Proof. In the statement of Lemma 35 take $u = v$, $w = e_i$. □

Corollary. Let u be a column of a symplectic elementary matrix, $v, w \in V$ such that $\langle u, v \rangle = \langle u, w \rangle = 0$, $a, b \in I$ such that $va = wb$. Then one has

$$Z(u, v, a, 0) = Z(u, w, b, 0).$$

Proof. Take $g \in \text{StSp}_{2l}(R)$ such that $\phi(g)u = e_i$. Then one has

$$\begin{aligned} Z(u, v, a, 0) &= {}^gZ(e_i, \phi(g)^{-1}v, a, 0) = {}^gY(e_i, \phi(g)^{-1}va, 0) = \\ &= {}^gY(e_i, \phi(g)^{-1}wb, 0) = {}^gZ(e_i, \phi(g)^{-1}w, b, 0) = Z(u, w, b, 0). \end{aligned}$$

□

Lemma 37. Consider $a \in I$, $r \in R$ and $u, v \in V$ such that $\langle u, v \rangle = 0$ and assume also that v is a column of a symplectic elementary matrix. Then

$$Z(u + vr, 0, a) = Z(u, 0, a)Z(v, 0, ar^2)Z(v, u, ar, 0).$$

Proof. Take $g \in \text{StSp}_{2l}(R)$ such that $\phi(g)v = e_i$. Then,

$${}^gZ(u + vr, 0, a) = Z(\phi(g)u + e_i r, 0, a).$$

Now, Lemma 35 (and Lemma 36) imply that

$$\begin{aligned} Z(\phi(g)u + e_i r, 0, a) &= Z(\phi(g)u, 0, a)Z(e_i r, 0, a)Y(e_i r, \phi(g)ua, 0) = \\ &= {}^g\left(Z(u, 0, a)Z(v, 0, ar^2)Z(v, u, ar, 0)\right). \end{aligned}$$

□

Lemma 38. Let $u \in V$ be column of a symplectic elementary matrix and let $v, w \in V$ be arbitrary columns such that $\langle u, v \rangle = \langle u, w \rangle = 0$, $a, b \in I$. Then

$$\begin{aligned} a) \quad &Z(u, v, a, 0)Z(u, w, a, 0) = Z(u, v + w, a, 0)Z(u, 0, a^2\langle v, w \rangle), \\ b) \quad &Z(u, v, a, 0)Z(u, v, b, 0) = Z(u, v, a + b, 0). \end{aligned}$$

Proof. Use the same trick as in Lemma 37. □

Definition. For $u, v \in V$ such that $\langle u, v \rangle = 0$, $a, b \in I$, set

$$Z(u, v, a, b) = Z(u, v, a, 0)Z(u, 0, b).$$

Lemma 39. Assume that u, u' are columns of symplectic elementary matrices, and let $v, v', w \in V$ be arbitrary columns such that $\langle u, v \rangle = \langle u, w \rangle = 0$,

and $\langle u', v' \rangle = 0$. Then for any $a, a', b, b' \in I, r \in R$ one has

- a) $Z(u, vr, a, b) = Z(u, v, ar, b)$,
- b) $Z(u, v, a, b)Z(u, w, a, c) = Z(u, v + w, a, b + c + a^2\langle v, w \rangle)$,
- c) $Z(u, v, a, 0)Z(u, v, b, 0) = Z(u, v, a + b, 0)$,
- d) $Z(u, v, a, 0) = Z(v, u, a, 0)$,
- e) $Z(u', v', a', b')Z(u, v, a, b)Z(u', v', a', b')^{-1} =$
 $= Z(T(u', v'a', b')u, T(u', v'a', b')v, a, b)$,
- f) $Z(u, u, a, 0) = Z(u, 0, 0, 2a)$,
- g) $Z(v + ur, 0, 0, a) = Z(v, 0, 0, a)Z(u, 0, 0, ar^2)Z(u, v, ar, 0)$.

Proof. For b) use Lemmas 38 and 31, the rest was already checked. \square

Definition. Let the relative symplectic van der Kallen group $\text{StSp}_{2l}^*(R, I)$ be the group defined by the set of generators

$$\{(u, v, a, b) \in V \times V \times I \times I \mid u \text{ is a column of} \\ \text{a symplectic elementary matrix, } \langle u, v \rangle = 0\}$$

and relations

$$(u, vr, a, b) = (u, v, ar, b) \text{ for any } r \in R, \quad (\text{T1})$$

$$(u, v, a, b)(u, w, a, c) = (u, v + w, a, b + c + a^2\langle v, w \rangle), \quad (\text{T2})$$

$$(u, v, a, 0)(u, v, b, 0) = (u, v, a + b, 0), \quad (\text{T3})$$

$$(u, v, a, 0) = (v, u, a, 0) \text{ for } v \text{ a column of} \\ \text{a symplectic elementary matrix,} \quad (\text{T4})$$

$$(u', v', a', b')(u, v, a, b)(u', v', a', b')^{-1} = \\ = (T(u', v'a', b')u, T(u', v'a', b')v, a, b), \quad (\text{T5})$$

$$(u, u, a, 0) = (u, 0, 0, 2a), \quad (\text{T6})$$

$$(u + vr, 0, 0, a) = (u, 0, 0, a)(v, 0, 0, ar^2)(v, u, ar, 0) \text{ for } v, u + vr \\ \text{also columns of symplectic elementary matrices.} \quad (\text{T7})$$

Remark. Clearly, Lemma 39 amounts to the existence of a homomorphism

$$\varpi: \text{StSp}_{2l}^*(R, I) \twoheadrightarrow \text{StSp}_{2l}(R, I),$$

sending (u, v, a, b) to $Z(u, v, a, b)$.

Lemma 40. Any triple $(u, v, a) \in V \times V \times R$ defines a homomorphism

$$\alpha_{u,v,a}: \text{StSp}_{2l}^*(R, I) \rightarrow \text{StSp}_{2l}^*(R, I)$$

sending generators (u', v', a', b') to $(T(u, v, a)u', T(u, v, a)v', a', b')$.

Proof. To show that $\alpha_{u,v,a}$ is well-defined we have to check that T1–T7 hold for the images of the generators. All of them are obvious, except T5, which is checked below.

$$\begin{aligned} & (T(u, v, a)u', T(u, v, a)v', a', b')(T(u, v, a)u'', T(u, v, a)v'', a'', b'') \cdot \\ & \quad \cdot (T(u, v, a)u', T(u, v, a)v', a', b')^{-1} = \\ & = (T(T(u, v, a)u', T(u, v, a)v'a', b')T(u, v, a)u'', \\ & \quad T(T(u, v, a)u', T(u, v, a)v'a', b')T(u, v, a)v'', a'', b'') = \\ & = (T(u, v, a)T(u', v', a')u'', T(u, v, a)T(u', v', a')v'', a'', b''). \end{aligned}$$

□

Lemma 41. There exists a well-defined homomorphism

$$\text{StSp}_{2l}(R) \rightarrow \text{Aut}(\text{StSp}_{2l}^*(R, I))$$

sending $X(u, v, a)$ to $\alpha_{u,v,a}$, i.e., absolute Steinberg group acts on relative van der Kallen group.

Proof. We need to verify that P1–P3 hold for $\alpha_{u,v,a}$. We check P1 below, P2 and P3 are left to the reader.

$$\begin{aligned} \alpha_{u,v,a}\alpha_{u,w,b}(u', v', a', b') &= \alpha_{u,v,a}(T(u, w, b)u', T(u, w, b)v', a', b') = \\ &= (T(u, v, a)T(u, w, b)u', T(u, v, a)T(u, w, b)v', a', b') = \\ &= (T(u, v+w, a+b+\langle v, w \rangle)u', T(u, v+w, a+b+\langle v, w \rangle)v', a', b') = \\ &= \alpha_{u,v+w,a+b+\langle v, w \rangle}(u', v', a', b'). \end{aligned}$$

□

Remark. Notice that ϖ preserves the action of $\text{StSp}_{2l}(R)$.

Definition. Set

$$\begin{aligned} ij(a) &= (e_{-j}, e_i, a\varepsilon_{-j}, 0) \text{ for } i \notin \{\pm j\}, \\ i, -i(a) &= (e_i, 0, 0, a). \end{aligned}$$

Lemma 42. *Steinberg relations KL0–KL7 hold for $_{ij}(a)$ and $_{i,-i}(a)$.*

Proof. To check KL0 use T4, for KL1 use T3 and T2. KL2 follows from the definition of the action. Below we verify the rest.

$$\begin{aligned}
(\text{KL3}) \llbracket X_{ij}(r), {}_{jk}(b) \rrbracket &= \\
&= (e_{-k}, T(e_i, e_{-j}r\varepsilon_{-j}, 0)e_j, b\varepsilon_{-k}, 0)(e_{-k}, e_j, b\varepsilon_{-k}, 0)^{-1} = \\
&= (e_{-k}, e_j + e_i r, b\varepsilon_{-k}, 0)(e_{-k}, -e_j, b\varepsilon_{-k}, 0) = \\
&= (e_{-k}, e_i r, b\varepsilon_{-k}, 0) = (e_{-k}, e_i, rb\varepsilon_{-k}, 0) = \\
&= (e_i, e_{-k}, rb\varepsilon_{-k}, 0) = {}_{ik}(rb);
\end{aligned}$$

$$\begin{aligned}
(\text{KL4}) \llbracket X_{i,-i}(r), {}_{-i,j}(b) \rrbracket &= \\
&= (e_{-j}, T(e_i, 0, r)e_{-i}, b\varepsilon_{-j}, 0)(e_{-j}, e_{-i}, b\varepsilon_{-j}, 0)^{-1} = \\
&= (e_{-j}, e_{-i} + e_i r\varepsilon_i, b\varepsilon_{-j}, 0)(e_{-j}, -e_{-i}, b\varepsilon_{-j}, 0) = \\
&= (e_{-j}, e_i r\varepsilon_i, b\varepsilon_{-j}, -rb^2) = \\
&= (e_{-j}, e_i r\varepsilon_i, b\varepsilon_{-j}, 0)(e_{-j}, 0, b\varepsilon_{-j}, -rb^2) = \\
&= (e_{-j}, e_i, rb\varepsilon_i\varepsilon_{-j}, 0)(e_{-j}, 0, 0, -rb^2) = {}_{ij}(rb\varepsilon_i) \cdot {}_{-j,j}(-rb^2);
\end{aligned}$$

$$\begin{aligned}
(\text{KL5}) \llbracket {}_{i,-i}(a), X_{-i,j}(s) \rrbracket &= \\
&= (e_i, 0, 0, a)(T(e_{-i}, e_{-j}s\varepsilon_{-j}, 0)e_i, 0, 0, -a) = \\
&= (e_i, 0, 0, a)(e_i + e_{-j}s\varepsilon_{-j}\varepsilon_{-i}, 0, 0, -a) = \\
&= (e_i, 0, 0, a)(e_i, 0, 0, -a)(e_{-j}, 0, 0, -as^2)(e_{-j}, e_i, -as\varepsilon_i\varepsilon_j, 0) = \\
&= (e_{-j}, e_i, as\varepsilon_i\varepsilon_{-j}, 0)(e_{-j}, 0, 0, -as^2) = {}_{ij}(as\varepsilon_i) \cdot {}_{-j,j}(-as^2);
\end{aligned}$$

$$\begin{aligned}
(\text{KL6}) \llbracket X_{ij}(r), {}_{j,-i}(b) \rrbracket &= \\
&= (e_i, T(e_i, e_{-j}r\varepsilon_{-j}, 0)e_j, b\varepsilon_i, 0)(e_i, -e_j, b\varepsilon_i, 0) = \\
&= (e_i, e_j + e_i r, b\varepsilon_i, 0)(e_i, -e_j, b\varepsilon_i, 0) = (e_i, e_i r, b\varepsilon_i, 0) = \\
&= (e_i, e_i, rb\varepsilon_i, 0) = (e_i, 0, 0, 2rb\varepsilon_i) = {}_{i,-i}(2rb\varepsilon_i).
\end{aligned}$$

The definition of the action together with T5 imply KL7. \square

Corollary. *There is a homomorphism*

$$\varrho: \text{StSp}_{2l}(R, I) \rightarrow \text{StSp}_{2l}^*(R, I)$$

sending $Y_{ij}(a)$ to $_{ij}(a)$ and preserving the action of $\text{StSp}_{2l}(R)$. Obviously, $\varpi\varrho = 1$, i.e. ϱ is a splitting for ϖ .

Lemma 43. For $v \in V$ such that $v_{-i} = 0$, set

$$\tilde{v}_- = \sum_{\substack{i \notin \{\pm j\} \\ i < 0}} e_i v_i \quad \text{and similarly} \quad \tilde{v}_+ = \sum_{\substack{i \notin \{\pm j\} \\ i > 0}} e_i v_i.$$

Then

$$\begin{aligned} (e_i, v, a, b) &= \\ &= {}_{i,-i}(b + 2av_i - a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) \prod_{\substack{i \notin \{\pm j\} \\ i < 0}} {}_{j,-i}(av_j \varepsilon_i) \prod_{\substack{i \notin \{\pm j\} \\ i > 0}} {}_{j,-i}(av_j \varepsilon_i). \end{aligned}$$

Proof.

$$\begin{aligned} (e_i, v, a, b) &= (e_i, 0, a, b)(e_i, v, a, 0) = \\ &= (e_i, 0, 0, b)(e_i, e_i v_i, a, 0)(e_i, \tilde{v}_- + \tilde{v}_+, a, 0) = \\ &= (e_i, 0, 0, b)(e_i, e_i, av_i, 0)(e_i, 0, a, -a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) \cdot \\ &\quad \cdot (e_i, \tilde{v}_- + \tilde{v}_+, a, a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) = \\ &= (e_i, 0, 0, b)(e_i, 0, 0, 2av_i)(e_i, 0, 0, -a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) \cdot \\ &\quad \cdot (e_i, \tilde{v}_-, a, 0)(e_i, \tilde{v}_+, a, 0) = \\ &= (e_i, 0, 0, b + 2av_i - a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) \cdot \\ &\quad \cdot \prod_{\substack{i \notin \{\pm j\} \\ i < 0}} (e_i, e_j v_j, a, 0) \prod_{\substack{i \notin \{\pm j\} \\ i > 0}} (e_i, e_j v_j, a, 0) = \\ &= {}_{i,-i}(b + 2av_i - a^2 \langle \tilde{v}_-, \tilde{v}_+ \rangle) \prod_{\substack{i \notin \{\pm j\} \\ i < 0}} {}_{j,-i}(av_j \varepsilon_i) \prod_{\substack{i \notin \{\pm j\} \\ i > 0}} {}_{j,-i}(av_j \varepsilon_i). \end{aligned}$$

□

Lemma 44.

$$\varrho \varpi = 1$$

Proof. Obviously, $\varrho \varpi({}_{ij}(a)) = {}_{ij}(a)$, then $\varrho \varpi((e_i, v, a, b)) = (e_i, v, a, b)$ by the previous lemma. For arbitrary column of symplectic elementary matrix u take $g \in \text{StSp}_{2l}(R)$ such that $\phi(g)e_i = u$ and use that ϱ and ϖ preserve action.

$$\begin{aligned} \varrho \varpi((u, v, a, b)) &= \varrho \varpi(g(e_i, \phi(g)^{-1}v, a, b)) = \\ &= {}^g \varrho \varpi((e_i, \phi(g)^{-1}v, a, b)) = {}^g(e_i, \phi(g)^{-1}v, a, b) = (u, v, a, b). \end{aligned}$$

□

References

- [1] van der Kallen W., “Another presentation for Steinberg groups”, *Indag. Math.*, **39**:4 (1977), 304–312.
- [2] Lavrenov A., “Another presentation for symplectic Steinberg groups”, arXiv.
- [3] Sinchuk S., *Parabolic factorizations of reductive groups*, Ph.D. Thesis, Saint-Petersburg State University, 2013.